Lectures 30 and 31

Classical Laminate Theory:

Classical Lamination theory (CLT) is a simple analysis through which the forces acting on a laminate are related to the mid-plane strains and curvatures which are assumed to be constant across the thickness. It is very useful in calculating stresses and strains in each lamina of the thin laminated structure.

Consider a laminate, of thickness h, consisting of laminas of varying thicknesses t_1, t_2 and so on. Let, the geometric mid-plane of the laminate contain the xy plane and the z axis defines the thickness direction. Let, the total number of laminae be N.

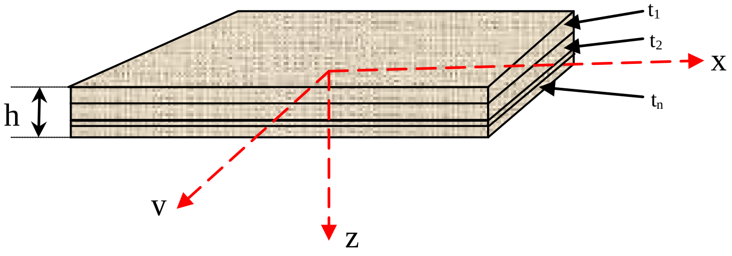

Figure 4.3 Coordinate system of Laminate

Assumptions:

- The laminate is thin and very wide
- Cross sections originally straight and perpendicular to the mid-plane remain straight and perpendicular to the mid-plane in the deformed state. i.e., $\gamma_{xz} = \gamma_{yz} = 0$
- Perfect bonding between the layers, i.e., there is no slippage.
- Strain distribution in the thickness direction is linear.
- All laminae are macroscopically homogeneous and behave in a linearly elastic manner.

Assume that a cross section shown in red colour in fig 4.2 originally straight and perpendicular to the mid-plane of the laminate also remains straight and perpendicular to the mid-plane in the deformed state. Further, assume that the point A at a distance z from the geometric mid-plane undergoes displacements u_o, v_o, and w_o along x, y, and z directions, respectively.

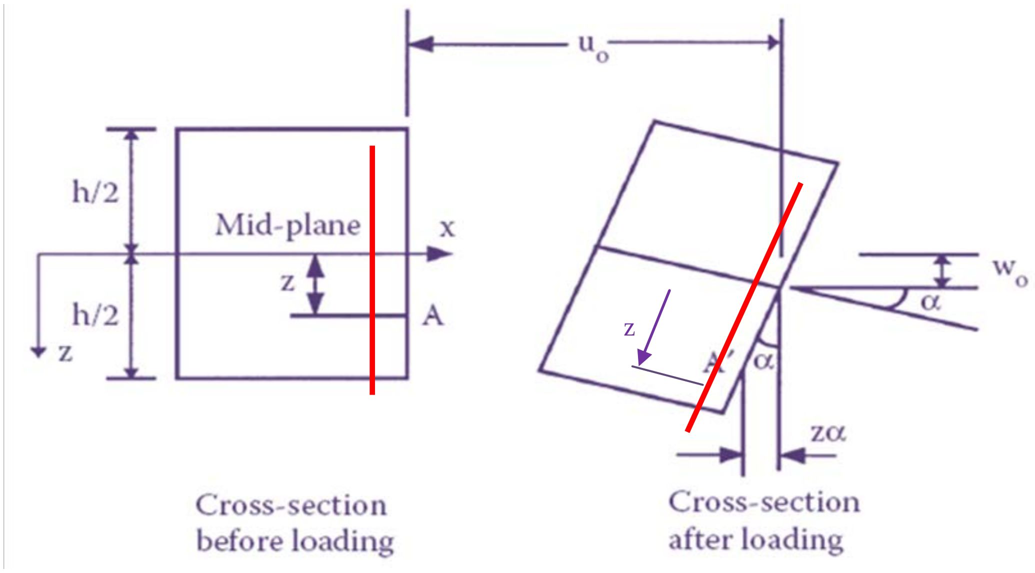

Figure 4.4 Deformation of the cross section of laminate in bending

The displacement u in the x direction of the point A is given by:

$$u = u_o - z\alpha \tag{4.4}$$

where,

$$\alpha = \frac{\partial w_o}{\partial x} \tag{4.5}$$

Therefore, the displacement *u* in the *x*-direction is :

$$u = u_o - z\frac{\partial w_o}{\partial x} \tag{4.6}$$

Similarly, taking a cross-section in the *y*–*z* plane would give the displacement in the *y*-direction as:

$$v = v_o - z\frac{\partial w_o}{\partial y} \tag{4.7}$$

The strains are defined by:

$$\varepsilon_x = \frac{\partial u}{\partial x} = \frac{\partial u_o}{\partial x} - z\frac{\partial^2 w_o}{\partial x^2} \tag{4.8}$$

$$\varepsilon_y = \frac{\partial v}{\partial y} = \frac{\partial v_o}{\partial y} - z\frac{\partial^2 w_o}{\partial y^2} \tag{4.9}$$

$$\gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} = \frac{\partial u_o}{\partial y} + \frac{\partial v_o}{\partial x} - 2z\frac{\partial^2 w_o}{\partial x\,\partial y} \tag{4.10}$$

The above strain displacement relations can be written as :

$$\begin{Bmatrix} \varepsilon_X \\ \varepsilon_y \\ \gamma_{Xy} \end{Bmatrix} = \begin{Bmatrix} \dfrac{\partial u_0}{\partial x} \\ \dfrac{\partial v_0}{\partial y} \\ \dfrac{\partial u_0}{\partial y} + \dfrac{\partial v_0}{\partial x} \end{Bmatrix} + z \begin{Bmatrix} -\dfrac{\partial^2 w_0}{\partial x^2} \\ -\dfrac{\partial^2 w_0}{\partial y^2} \\ -2\dfrac{\partial^2 w_0}{\partial x \partial y} \end{Bmatrix} \tag{4.11}$$

$$\begin{Bmatrix} \varepsilon_X \\ \varepsilon_y \\ \gamma_{Xy} \end{Bmatrix} = \begin{Bmatrix} \varepsilon_x^0 \\ \varepsilon_y^0 \\ \gamma_{Xy}^0 \end{Bmatrix} + z \begin{Bmatrix} k_x \\ k_y \\ k_{xy} \end{Bmatrix} \tag{4.12}$$

or

$$\{\varepsilon\} = \{\varepsilon_0\} + z\,\{k\} \tag{4.13}$$

where, $$\varepsilon_x^0 = \frac{\partial u_o}{\partial x} \quad ; \quad \varepsilon_y^0 = \frac{\partial vo}{\partial y} \quad ; \quad \gamma_{xy}^0 = \frac{\partial u_o}{\partial y} + \frac{\partial v_o}{\partial x} \tag{4.14}$$

$$k_x = -\frac{\partial^2 w_o}{\partial x^2} \; ; \quad k_y = -\frac{\partial^2 w_o}{\partial y^2} \; ; \quad k_{xy} = -2\frac{\partial^2 w_o}{\partial x\, \partial y} \tag{4.15}$$

The above equations indicate that the strains in a laminate vary linearly across the thickness, which reinforce the assumptions made earlier. Stresses in any lamina can be determined using the stress-strain relation for the lamina.

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix} = \begin{bmatrix} \bar{Q}_{11} & \bar{Q}_{12} & \bar{Q}_{16} \\ \bar{Q}_{12} & \bar{Q}_{22} & \bar{Q}_{26} \\ \bar{Q}_{16} & \bar{Q}_{26} & \bar{Q}_{66} \end{bmatrix} \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix} \tag{4.16}$$

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix} = \begin{bmatrix} \bar{Q}_{11} & \bar{Q}_{12} & \bar{Q}_{16} \\ \bar{Q}_{12} & \bar{Q}_{22} & \bar{Q}_{26} \\ \bar{Q}_{16} & \bar{Q}_{26} & \bar{Q}_{66} \end{bmatrix}_k \begin{Bmatrix} \varepsilon_x^0 \\ \varepsilon_y^0 \\ \gamma_{xy}^0 \end{Bmatrix} + z \begin{bmatrix} \bar{Q}_{11} & \bar{Q}_{12} & \bar{Q}_{16} \\ \bar{Q}_{12} & \bar{Q}_{22} & \bar{Q}_{26} \\ \bar{Q}_{16} & \bar{Q}_{26} & \bar{Q}_{66} \end{bmatrix} \begin{Bmatrix} k_x \\ k_y \\ k_{xy} \end{Bmatrix} \tag{4.17}$$

Thus, the variation of stress through the laminate thickness is obtained by calculating the stress variations in all the laminae. From the above expressions it is clear that the strains will be continuous across the thickness whereas, the stresses will be discontinuous across the thickness as the material properties will change for each layer, depending upon the orientation.

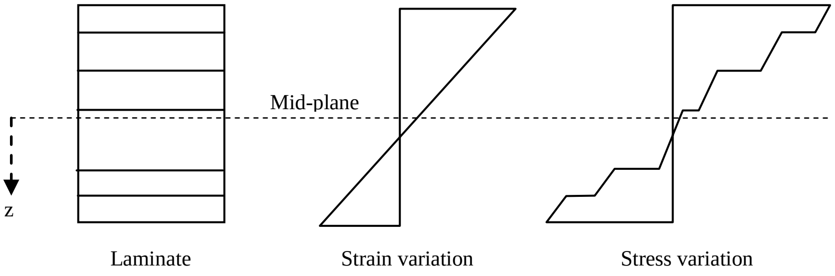

Figure 4.5 Stress strain variation in laminate

Force and Moment Resultants:

The stresses in a laminate vary from layer to layer. Hence, it is convenient to deal with a simpler but, equivalent system of forces and moments acting on the laminate cross section.

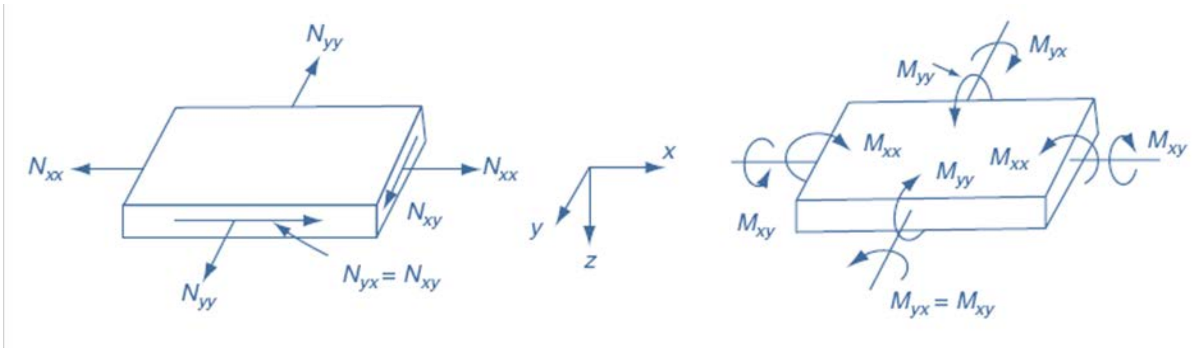

Figure 4.6 Resultant forces and moments on a laminate

Consider a laminate which is subjected to forces and moments as shown in figure 4.6. The forces and moments are called as resultant forces and resultant moments respectively and defined per unit width of the laminate and acting at the mid-plane of the laminate.

Resultant forces are obtained by integrating the corresponding stresses through the laminate thickness, h. Thus,

$$N_X = \int_{-h/2}^{h/2} \sigma_x \, dz \tag{4.18}$$

$$N_y = \int_{-h/2}^{h/2} \sigma_y \, dz \tag{4.19}$$

$$N_{Xy} = \int_{-h/2}^{h/2} \tau_{xy}\, dz \tag{4.20}$$

Similarly, the resultant moments are obtained by integrating the corresponding stresses times the moment arm with respect to the mid-plane through the thickness of laminate. The expressions for the moment resultants are given by :

$$M_X = \int_{-h/2}^{h/2} \sigma_x\, z\, dz \tag{4.21}$$

$$M_y = \int_{-h/2}^{h/2} \sigma_y\, z\, dz \tag{4.22}$$

$$M_{Xy} = \int_{-h/2}^{h/2} \tau_{xy}\, z\, dz \tag{4.23}$$

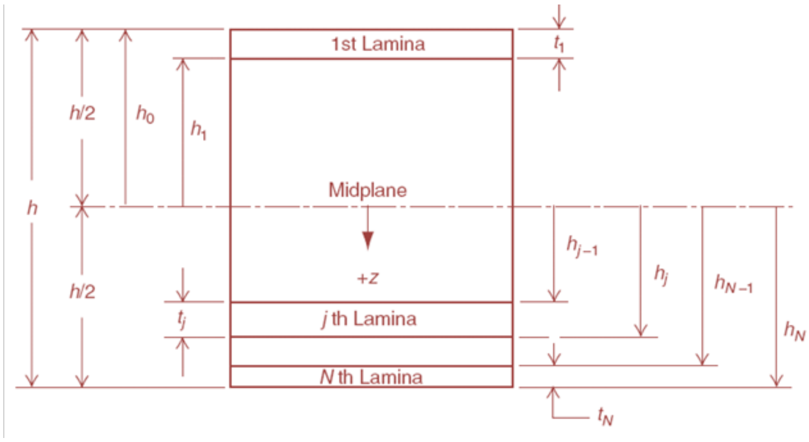

Figure 4.7 laminate with N laminas

Consider a laminate consisting of N orthotropic laminae as shown in Figure 4.7. As the number of layers is a finite one, the force-moment system acting at the midplane of the laminate can be obtained by replacing the continuous integral by the summation of integrals of each lamina. Therefore,

$$\begin{Bmatrix} N_X \\ N_y \\ N_{Xy} \end{Bmatrix} = \int_{-h/2}^{h/2} \begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix} dz = \sum_{k=1}^{n} \int_{z_{k-1}}^{z_k} \begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix} dz \tag{4.24}$$

$$\begin{Bmatrix} M_X \\ M_y \\ M_{Xy} \end{Bmatrix} = \int_{-h/2}^{h/2} \begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix} z\, dz = \sum_{k=1}^{n} \int_{z_{k-1}}^{z_k} \begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix} z\, dz \tag{4.25}$$

The stress - strain relation for an arbitrary layer is given by,

$$\begin{Bmatrix} \sigma_X \\ \sigma_y \\ \tau_{Xy} \end{Bmatrix} = \begin{bmatrix} \bar{Q}_{11} & \bar{Q}_{12} & \bar{Q}_{16} \\ \bar{Q}_{12} & \bar{Q}_{22} & \bar{Q}_{26} \\ \bar{Q}_{16} & \bar{Q}_{26} & \bar{Q}_{66} \end{bmatrix} \begin{Bmatrix} \varepsilon_X \\ \varepsilon_y \\ \gamma_{Xy} \end{Bmatrix} = [\bar{Q}_{ij}]\{\{\varepsilon_o\} + z\,\{k\}\} \tag{4.26}$$

Thus the equations for the resultant forces and moments become,

$$\begin{Bmatrix} N_X \\ N_y \\ N_{Xy} \end{Bmatrix} = \sum_{k=1}^{n} \int_{z_{k-1}}^{z_k} [\bar{Q}_{ij}]_k \ \{\{\varepsilon_o\} + z\,\{k\}\}\, dz \tag{4.27}$$

$$\{N\} = \sum_{k=1}^{n} [\bar{Q}_{ij}]_k \ [z_k - z_{k-1}]\{\varepsilon_o\} + \frac{1}{2}\sum_{k=1}^{n} [\bar{Q}_{ij}]_k \ [z_k^2 - z_{k-1}^2]\{k\}$$

$$\{N\} = [A]\{\varepsilon_o\} + [B]\{k\} \tag{4.28}$$

$$\begin{Bmatrix} M_X \\ M_y \\ M_{Xy} \end{Bmatrix} = \sum_{k=1}^{n} \int_{z_{k-1}}^{z_k} [\bar{Q}_{ij}]_k \{\{\varepsilon_o\} + z\,\{k\}\}\, z\, dz \tag{4.29}$$

$$\begin{Bmatrix} M_X \\ M_y \\ M_{Xy} \end{Bmatrix} = \sum_{k=1}^{n} [\bar{Q}_{ij}]_k \int_{z_{k-1}}^{z_k} \{\varepsilon_o\} z\, dz + \sum_{k=1}^{n} [\bar{Q}_{ij}]_k \int_{z_{k-1}}^{z_k} \{k\} z^2\, dz \tag{4.30}$$

$$\{M\} = \sum_{k=1}^{n} [\bar{Q}_{ij}]_k \frac{1}{2}[z_k^2 - z_{k-1}^2]\{\varepsilon_o\} + \sum_{k=1}^{n} [\bar{Q}_{ij}]_k \ \frac{1}{3}[z_k^3 - z_{k-1}^3]\{k\}$$

$$\{M\} = [B]\{\varepsilon_o\} + [D]\{k\} \tag{4.31}$$

or, $$\begin{Bmatrix} N \\ M \end{Bmatrix} = \begin{bmatrix} A & B \\ B & D \end{bmatrix} \begin{Bmatrix} \varepsilon_o \\ k \end{Bmatrix}$$

where, [A] - Extensional Stiffness matrix

[B] - Extensional bending coupling stiffness matrix

[D] - Bending stiffness matrix

$$[A] = \sum_{k=1}^{n} \left[\overline{Q}_{ij}\right]_k \left[z_k - z_{k-1}\right] = \begin{bmatrix} A_{11} & A_{12} & A_{16} \\ A_{12} & A_{22} & A_{26} \\ A_{16} & A_{26} & A_{66} \end{bmatrix} \quad (4.32)$$

$$[A] = \sum_{k=1}^{n} \left[\overline{Q}_{ij}\right]_k \left[z_k^2 - z_{k-1}^2\right] = \begin{bmatrix} B_{11} & B_{12} & B_{16} \\ B_{12} & B_{22} & B_{26} \\ B_{16} & B_{26} & B_{66} \end{bmatrix} \quad (4.33)$$

$$[D] = \frac{1}{3}\sum_{k=1}^{n} \left[\overline{Q}_{ij}\right]_k \left[z_k^3 - z_{k-1}^3\right] = \begin{bmatrix} D_{11} & D_{12} & D_{16} \\ D_{12} & D_{22} & D_{26} \\ D_{16} & D_{26} & D_{66} \end{bmatrix} \quad (4.34)$$

So, the resultant forces and moments are thus related to mid-plane strains and curvatures.

$$\begin{Bmatrix} N_x \\ N_y \\ N_{xy} \end{Bmatrix} = \begin{bmatrix} A_{11} & A_{12} & A_{16} \\ A_{12} & A_{22} & A_{26} \\ A_{16} & A_{26} & A_{66} \end{bmatrix} \begin{Bmatrix} \varepsilon_x^o \\ \varepsilon_y^o \\ \gamma_{xy}^o \end{Bmatrix} + \begin{bmatrix} B_{11} & B_{12} & B_{16} \\ B_{12} & B_{22} & B_{26} \\ B_{16} & B_{26} & B_{66} \end{bmatrix} \begin{Bmatrix} k_x \\ k_y \\ k_{xy} \end{Bmatrix} \quad (4.35)$$

$$\begin{Bmatrix} M_x \\ M_y \\ M_{xy} \end{Bmatrix} = \begin{bmatrix} B_{11} & B_{12} & B_{16} \\ B_{12} & B_{22} & B_{26} \\ B_{16} & B_{26} & B_{66} \end{bmatrix} \begin{Bmatrix} \varepsilon_x^o \\ \varepsilon_y^o \\ \gamma_{xy}^o \end{Bmatrix} + \begin{bmatrix} D_{11} & D_{12} & D_{16} \\ D_{12} & D_{22} & D_{26} \\ D_{16} & D_{26} & D_{66} \end{bmatrix} \begin{Bmatrix} k_x \\ k_y \\ k_{xy} \end{Bmatrix} \quad (4.36)$$

[A} - matrix is called extensional stiffness matrix

[B]- matrix is called extension-bending coupling stiffness matrix

[D]- matrix is called bending stiffness matrix

Comments on A,B, and D matrices:

(i). A_{16} and A_{26} are the coupling terms which relate in-plane normal forces to mid-plane shear strain and in-plane shear force to mid-plane normal strains.

(ii). B_{11}, B_{12} and B_{22} couple in-plane normal forces to bending curvatures and bending moments to mid-plane normal strains.

(iii). B_{16} and B_{26} couple in-plane normal forces to twisting curvature and twisting moment

to mid-plane normal strains.

(iv). B_{66} couples in-plane shear force to twisting curvature and twisting moment to mid-plane shear strain.

(v). D_{16} and D_{26} couple bending moments to twisting curvature and twisting moment to bending curvatures.

The couplings between normal forces and shear strains , bending moments and twisting curvatures and so on, occur only in laminated structures and not in a monolithic structure. If the laminate is properly constructed, some of these couplings can be eliminated.

Based on the nature of laminate layers and their fiber orientations, there exist some influences on these A, B, and D matrices.

Assignments:

1) Find the A,B and D matrices for the following laminates.

a) $[0^o/30^o/60^o/90^o]$

b) $[0^o/45^o/-45^o/90^o]$

c)$[0^o/30^o/45^o]_{s.}$

Thickness of each layer is 1mm and the material properties of the layer are :

$E_1 = 147$GPa, $E_2 = 15$GPa, $G_{12} = 12$GPa and $\nu_{12} = 0.3$.

Reference:

"Mechanics of Composite Structural Elements", H Altenbach, J Altenbach and W Kissing, Springer publications.

" Principles of Composite Material Mechanics", Ronald F Gibson, CRC Press.

Lecture 32

A matrix:

A matrix is called as the extensional stiffness matrix. In order to have a laminate with no coupling between the normal stresses and shear strain, A_{16} and A_{26} must be zero. The contribution of one lamina to a term A_{ij} can be nullified by another lamina of the same thickness but opposite sign of $\bar{Q}_{ij}$ term.

$\bar{Q}_{11}$, $\bar{Q}_{12}$, $\bar{Q}_{22}$, and $\bar{Q}_{66}$ are always positive and greater than zero. So, A_{11}, A_{22}, A_{12} and A_{66} cannot be made equal to zero. $\bar{Q}_{16}$ and $\bar{Q}_{26}$ are zero for orientations of 0^o or 90^o and can be positive or negative for intermediate values. Since, $\bar{Q}_{16}$ and $\bar{Q}_{26}$ are odd functions of θ,

$$[\ \bar{Q}_{16} = (\bar{Q}_{11} - \bar{Q}_{12} - 2 * \bar{Q}_{66})\cos^3\theta \sin\theta - (\bar{Q}_{22} - \bar{Q}_{12} - 2 * \bar{Q}_{66}) \cos\theta \sin^3\theta$$

$$\bar{Q}_{26} = (\bar{Q}_{11} - \bar{Q}_{12} - 2 * \bar{Q}_{66})\cos\theta \sin^3\theta - (\bar{Q}_{22} - \bar{Q}_{12} - 2 * \bar{Q}_{66}) \cos^3\theta \sin\theta\ ] \qquad (4.37)$$

For equal positive and negative orientations they are equal in magnitude but opposite in sign.

Therefore, A_{16} and A_{26} can be made equal to zero if for every lamina oriented at a positive angle θ in the laminate there exists another lamina of equal thickness and oriented at the equal negative angle θ. Relative positions of the two laminae are immaterial.

The following forms of laminate has the terms A_{16} and A_{26} being equal to zero.

(i) Cross-ply laminate with laminae oriented at 0^o or 90^o only.

(i) Angle-ply laminate with equal number of laminae oriented at ± θ angle.

B matrix:

The B matrix is called as the extension-bending coupling stuffiness matrix. The contribution of a lamina to a particular term of the B matrix is given by the product of the corresponding term in the $\bar{Q}$ matrix and the difference of the squares of z coordinates of the top and bottom surface of each ply. The contribution of a lamina above the geometric mid-plane can be nullified by placing an identical lamina below the mid-plane. This leads to $[B_{ij}] = 0$.

All symmetric laminates have $[B_{ij}] = 0$.

By controlling the orientations of the laminate all the terms in the B matrix can be made to zero or some terms can be made to zero if the laminate is not symmetric. In a cross ply laminate, the extension- twisting coupling terms B_{16} and B_{26} can be made equal to zero. In an anti-symmetric cross ply laminate in addition to B_{16} and B_{26} , B_{12} can also be made to zero.

D matrix:

D matrix is called as the bending stiffness matrix. The geometric contribution $(h_k^3 - h_{k-1}^3)$ in the definition of D matrix is always positive. So, D_{11}, D_{12}, D_{22}, and D_{66} are always positive. Since, $\bar{Q}_{16}$ and $\bar{Q}_{26}$ are odd functions of θ, D_{16} and D_{26} can be made equal to zero.

The following forms of laminate has the terms D_{16} and D_{26} being equal to zero.

(i)If all the laminae are oriented at 0^o or 90^o

(ii)If for every lamina oriented at a positive angle θ above the mid-plane there exists an identical lamina placed at an equal distance below the mid-plane but negative θ, there will not be any symmetry about mid-plane leading to the situation of having non-zero $[B_{ij}]$ matrix.

(i) Single specially orthotropic layer:

The laminate contains only one layer so that the laminate itself is considered as symmetric about its mid-plane. As, it is a specially orthotropic, $\bar{Q}_{16} = \bar{Q}_{26} = 0$. Therefore, zero elements in A, B, and D matrices are:

$$[B_{ij}] = 0; \qquad A_{16} = A_{26} = 0; \qquad D_{16} = D_{26} = 0$$

The resultant forces depend only on the in-plane strains and the resultant moments depend only on the curvatures.

(ii) Single generally orthotropic layer:

Since, it is a symmetric layer, $[B_{ij}] = 0$.

Therefore, no coupling between bending and extension exists. Extensional forces depend on shearing strain as well as on extensional strain. The resulting shearing force will produce extensional strains and shear strains. Moment resultants will produce curvatures and twist.

(iii) Single anisotropic layer:

Since, it is a symmetric layer, $[B_{ij}] = 0$.

The only difference between anisotropic and generally orthotropic is the formation of the stiffness matrix. In the case of anisotropic, the stiffness matrix is formed directly from reduced stiffness matrix, i.e.

$$[Q] = \begin{bmatrix} Q_{11} & Q_{12} & Q_{16} \\ Q_{12} & Q_{22} & Q_{26} \\ Q_{16} & Q_{26} & Q_{66} \end{bmatrix} \qquad (4.38)$$

where, as in the case of generally orthotropic, the stiffness matrix is formed from tranformed reduced stiffness matrix. i.e.

$$[\bar{Q}] = \begin{bmatrix} \bar{Q}_{11} & \bar{Q}_{12} & \bar{Q}_{16} \\ \bar{Q}_{12} & \bar{Q}_{22} & \bar{Q}_{26} \\ \bar{Q}_{16} & \bar{Q}_{26} & \bar{Q}_{66} \end{bmatrix} \qquad (4.39)$$

In an anisotropic layer the number of non zero independent constants will be higher than that in generally orthotropic layer.

(iv) Symmetric laminate with multiple isotropic layers:

Since, it is a symmetric laminate, $[B_{ij}] = 0$. As $\bar{Q}_{16} = \bar{Q}_{26} = 0$, $A_{16} = A_{26} = 0$ and $D_{16} = D_{26} = 0$. Moreover, due to isotropic, $A_{11} = A_{22}$ and $D_{11} = D_{22}$.

(v) Symmetric laminate with multiple specially orthotropic layers:

Since, it is a symmetric laminate, $[B_{ij}] = 0$. As $\bar{Q}_{16} = \bar{Q}_{26} = 0$, $A_{16} = A_{26} = 0$ and $D_{16} = D_{26} = 0$.

(vi) Symmetric laminate with multiple generally orthotropic layers:

Since, it is a symmetric laminate, $[B_{ij}] = 0$.

(vii) Anti-symmetric cross-ply laminates:

Since, $\bar{Q}_{16} = \bar{Q}_{26} = 0$ due to cross-ply, $A_{16} = A_{26} = 0$ and $D_{16} = D_{26} = 0$. As the laminate consists of an even number of laminae, $[B_{ij}] = 0$ except the terms B_{11} and B_{22}. If the number of layers increases, the coupling stiffness B_{11} can be made to approach zero.

$$\begin{Bmatrix} N_x \\ N_y \\ N_{xy} \\ M_x \\ M_y \\ M_{xy} \end{Bmatrix} = \begin{bmatrix} A_{11} & A_{12} & 0 & B_{11} & 0 & 0 \\ A_{12} & A_{22} & 0 & 0 & -B_{11} & 0 \\ 0 & 0 & A_{66} & 0 & 0 & 0 \\ B_{11} & 0 & 0 & D_{11} & D_{12} & 0 \\ 0 & -B_{11} & 0 & D_{12} & D_{22} & 0 \\ 0 & 0 & 0 & 0 & 0 & D_{66} \end{bmatrix} \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \\ k_x \\ k_y \\ k_{xy} \end{Bmatrix} \tag{4.40}$$

(viii) Anti-symmetric angle-ply laminates:

Due to anti-symmetric angle-ply, $A_{16} = A_{26} = 0$ and $D_{16} = D_{26} = 0$. As the laminate consists of an even number of laminae, $[B_{ij}] = 0$ except the terms B_{16} and B_{26}. If the number of layers increases, the coupling stiffness B_{16} and B_{26} can be made to approach zero.

(ix) Non-symmetric laminates

Specially orthotropic:

Since, $\bar{Q}_{16} = \bar{Q}_{26} = 0$, $A_{16} = A_{26} = 0$, $B_{16} = B_{26} = 0$, and $D_{16} = D_{26} = 0$.

Extension-shear coupling, normal-twisting coupling and bending twisting coupling terms will be eliminated.

Generally orthotropic:

It has all the terms in A, B, and D matrices. All the terms in the A, B and D matrices will be existing and all the coupling effects will be felt by the laminate.

Problems:

Thickness of each layer is 1mm and the material properties of the layer are

E_{11} = 147GPa, E_{22} = 15GPa, G_{12} = 12GPa and ν_{12} = 0.3 . Find the A, B and D matrices for the following cases.

a) A single layer, with 0^o , orientation.

b) A single layer with orientation 45^o

c) $0^o/90^o/90^o/0^o$

d) $0^o/30^o/45^o/45^o/30^o/0^o$

e) $30^o/45^o/60^o$

Identify the type of the above laminates.

Reference:

"Mechanics of Composite Structural Elements", H Altenbach, J Altenbach and W Kissing, Springer publications.

" Principles of Composite Material Mechanics", Ronald F Gibson, CRC Press.

Lecture 33

(x) Quasi – isotropic laminate:

In a quasi-isotropic laminate, the extensional stiffness matrix [A] is isotropic. So, the following relations are valid.

$$A_{11} = A_{22} \qquad A_{66} = \frac{\left(A_{11} - A_{12}\right)}{2} \qquad A_{16} = A_{26} = 0$$

$$[A] = \begin{bmatrix} \dfrac{E_{11}.t}{1-\nu_{12}^2} & \dfrac{\nu\, E_{11}.t}{1-\nu_{12}^2} & 0 \\ \dfrac{\nu_{12} E_{11}\, t}{1-\nu_{12}^2} & \dfrac{E_{11}\, t}{1-\nu_{12}^2} & 0 \\ 0 & 0 & \dfrac{E_{11}.t}{2\left(1+\nu_{12}\right)} \end{bmatrix} \tag{4.41}$$

The conditions for a laminate to be quasi-isotropic are:

(i) The total number of layers must be three or more.

(ii) The individual layers must have identical stiffness matrices [Q] and thicknesses.

(iii) The layers must be oriented at equal incremental angles. If the total number of layers is N, the angle between two adjacent layers has to be π/N.

Laminates of $[0^o/60^o/30^o]$ and $[0^o/45^o/90^o/-45^o]$ are quasi-isotropic.

Determination of mid-plane strains and curvatures:

In a composite laminate when it is subjected to external loads, strains are produced and there will be deflections. In order to obtain the deformations mid plane strains and curvatures are important. Hence, it is important that the strains are written in terms of the applied loads by using the different coefficient matrices. The relation between the resultant forces and moments and the mid-plane strains and the mid-plane curvatures is established as :

$$\begin{Bmatrix} \mathbf{N} \\ \mathbf{M} \end{Bmatrix} = \begin{bmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{B} & \mathbf{D} \end{bmatrix} \begin{Bmatrix} \varepsilon_o \\ \mathbf{k} \end{Bmatrix} \tag{4.42}$$

or

$$\{N\} = [A]\{\varepsilon_o\} + [B]\{k\} \tag{4.43}$$

$$\{M\} = [B]\{\varepsilon_o\} + [D]\{k\} \tag{4.44}$$

Equation (4.1) can be rewritten for mid-plane strains as:

$$\{\varepsilon_o\} = [A]^{-1}\{N\} - [A]^{-1}[B]\{k\} \tag{4.45}$$

Substituting Eqn. (4.3) in Eqn. (4.2)

$$\{M\} = [B][A]^{-1}\{N\} - \{[B][A]^{-1}[B] - [D]\}\{k\} \tag{4.46}$$

Combining Eqns. (4.3) and (4.4) and writing in matrix form, we get :

$$\begin{Bmatrix} \varepsilon_o \\ M \end{Bmatrix} = \begin{bmatrix} A^* & B^* \\ B^* & D^* \end{bmatrix} \begin{Bmatrix} N \\ k \end{Bmatrix} \tag{4.47}$$

Eqn.(4.5) is called partially inverted form of constitutive relations.

where,

$$[A^*] = [A]^{-1}$$

$$[B^*] = - [A]^{-1}[B]$$

$$[C^*] = [B]\ [A]^{-1} = - [B^*]^T$$

$$[D^*] = [D] - [B]\ [A]^{-1}[B]$$

Equation (4.5) can be rewritten as :

$$\{\varepsilon_o\} = [A^*]\{N\} + [B^*]\{k\} \tag{4.48}$$

$$\{M\} = [C^*]\{N\} + [D^*]\{k\} \tag{4.49}$$

From the Equation (4.7), the expression for [k] is :

$$\{k\} = [D^*]^{-1}\{M\} - [D^*]^{-1}[C^*]\{N\} \tag{4.50}$$

Substituting equation (4.8) in the equation (4.6), we get :

$$\{\varepsilon_o\} = \{[A^*] - [B^*][D^*]^{-1}[C^*]\}\{N\} + [B^*][D^*]^{-1}\{k\} \tag{4.51}$$

Equations (4.8) and (4.9) can be combined to obtain a fully inverted form of constitutive relations:

$$\begin{Bmatrix} \varepsilon_o \\ \mathbf{k} \end{Bmatrix} = \begin{bmatrix} \mathbf{A}' & \mathbf{B}' \\ \mathbf{C}' & \mathbf{D}' \end{bmatrix} \begin{Bmatrix} \mathbf{N} \\ \mathbf{M} \end{Bmatrix} \tag{4.52}$$

where,

$$[A'] = [A^*] - [B^*]\,[D^*]^{-1}\,[C^*]$$

$$[B'] = [B^*]\,[D^*]^{-1}$$

$$[C'] = -\,[D^*]^{-1}\,[C^*]$$

$$= [D^*]^{-1}\,[B^*]^T$$

$$= [B^1]$$

$$[D'] = [B^*]^{-1} \qquad (4.53)$$

By knowing the external loads and moments the strains can be obtained from which the displacements and stresses can be obtained.

Problems:

Thickness of each layer is 1mm and the material properties of the layer are

a) E_{11} = 147 GPa, E_{12} = 15 GPa, G_{12} = 12 GPa and ν_{12} = 0.3 . For the laminate with orientations $[45^o/-45^o]$, find the lamina stresses due to the load of Nxx = 100 kN/m.

b) Verify: $[0^o/60^o/-60^o]$ and $[0^o/45^o/90^o/-45^o]$ are quasi-isotropic.

Reference:

"Mechanics of Composite Structural Elements", H Altenbach, J Altenbach and W Kissing, Springer publications.

" Principles of Composite Material Mechanics", Ronald F Gibson, CRC Press.

"Fiber-Reinforced Composites", P K Mallick, CRC Press.

Lectures 34 -37

Determination of stresses and strains:

Using CLT the resultant forces and moments can be calculated if the mid-plane strains as well as curvatures are known with the known stiffness matrices of each lamina. Similarly, the mid-plane strains and curvatures can also be calculated for any set of applied resultant forces and moments from the equation (4.10).

Stresses and strains in any lamina can be calculated by the following procedure:

i. Calculate stiffness matrices for each lamina

ii. Calculate A,B, and D matrices for the laminate

iii. Calculate A', B', and D' matrices

iv. Calculate mid-plane strains and curvatures for the laminate by using the Eqn.(4.10).

v. Calculate in-plane strains ε_{xx}, ε_{yy} , and γ_{xy} at any location of the laminate by using the relation

$$\begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix} = \begin{Bmatrix} \varepsilon_x^o \\ \varepsilon_y^o \\ \gamma_{xy}^o \end{Bmatrix} + z \begin{Bmatrix} k_x \\ k_y \\ k_{xy} \end{Bmatrix} \tag{4.54}$$

vi. Calculate in-plane stresses σ_{xx}, σ_{yy} , and τ_{xy} for each lamina by using the expression

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix} = \begin{bmatrix} \bar{Q}_{11} & \bar{Q}_{12} & \bar{Q}_{16} \\ \bar{Q}_{12} & \bar{Q}_{22} & \bar{Q}_{26} \\ \bar{Q}_{16} & \bar{Q}_{26} & \bar{Q}_{66} \end{bmatrix} \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix} \tag{4.55}$$

From the above expressions it will be clear that the stresses across the thickness are discontinuous whereas the strains are continuous.

Problems (Module IV):

Problem 4.1: Find A, B, and D matrices for the 2-ply laminate as shown in the figure. Assume both the laminae have identical stiffness matrix Q as follows:

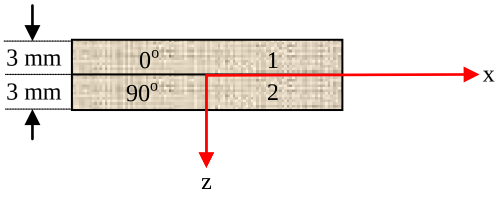

Fig.(i): Figure showing the laminate orientation

$$[Q] = \begin{bmatrix} 130 & 2.5 & 0 \\ 2.5 & 10 & 0 \\ 0 & 0 & 3.5 \end{bmatrix} GPa$$

Solution:

As the laminate is not symmetric about its mid-plane, it will have all the A,B, and D matrices.

First find out the stiffness matrices corresponding to each lamina of (0^o) and (90^o).

$$\left[\bar{Q}\right]0 = [Q] = \begin{bmatrix} 13 & 2.5 & 0 \\ 2.5 & 1 & 0 \\ 0 & 0 & 3.5 \end{bmatrix} \text{GPa} \qquad (4.56)$$

Using transformation,

$$\left[\bar{Q}\right]90 = \begin{bmatrix} 1 & 2.5 & 0 \\ 2.5 & 13 & 0 \\ 0 & 0 & 3.5 \end{bmatrix} GPa \qquad (4.57)$$

The laminate has two layers with orientations of 0^o and 90^o. The values of h_0, h_1 and h_2 are as shown in figure.

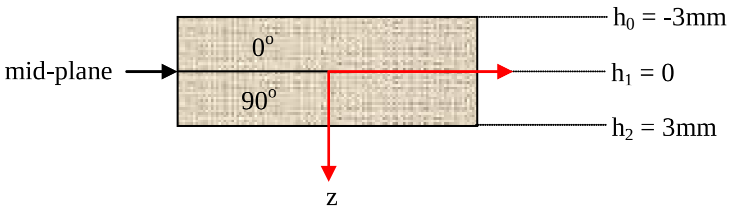

Fig.(ii) showing the laminate orientation

[A] matrix is determined by using the equation,

$$\left[A_{ij}\right] = \sum_{k=1}^{n} \left(\bar{Q}_{ij}\right)_k \left(h_k - h_k - 1\right) \qquad n = 2 \text{ layers}$$

$$= (\bar{Q}_{ij})_{0^o}[0 - (-3)] + (\bar{Q}_{ij})_{90^o}[3 - 0]$$

$$= 3*(\bar{Q}_{ij})_{0^o} + 3*(\bar{Q}_{ij})_{90^o}$$

Hence, $A_{11} = 3 * ((\bar{Q}_{11})_{0^o} + (\bar{Q}_{11})_{90^o}) = 3 * (130 + 10) = 420$

Thus,

$$[A] = \begin{bmatrix} 42 & 15 & 0 \\ 15 & 42 & 0 \\ 0 & 0 & 21 \end{bmatrix} \frac{\times 10^9\ N}{m} \tag{4.58}$$

[B] matrix is determined by using the equation,

$$\left[B_{ij}\right] = \frac{1}{2} \sum_{k=1}^{n} \left(\bar{Q}_{ij}\right)_k \left(h_k^2 - h_{k-1}^2\right) \tag{4.59}$$

$$= 0.5 *\{ (\bar{Q}_{ij})_{0^o}[0 - (-3)^2] + (\bar{Q}_{ij})_{90^o}[3^2 - 0] \}$$

$$= -4.5 *(\bar{Q}_{ij})_{0^o} + 4.5 *(\bar{Q}_{ij})_{90^o} = 4.5 * ((\bar{Q}_{ij})_{90^o} - (\bar{Q}_{ij})_{0^o})$$

Hence, $B_{11} = 4.5 * (10 - 130) = -540$.

Thus,

$$[B] = \begin{bmatrix} -54 & 0 & 0 \\ 0 & 54 & 0 \\ 0 & 0 & 0 \end{bmatrix} \times 10^9 \text{ N} \qquad (4.60)$$

[D] matrix is determined by using the equation,

$$\left[D_{ij}\right] = \frac{1}{3}\sum_{k=1}^{n}\left(\bar{Q}_{ij}\right)_k \left(h_{3(k)} - h_{3(k-1)}\right) \qquad (4.61)$$

$$= \tfrac{1}{3} \{ (\bar{Q}_{ij})_{0^o}[0 - (-3)^3] + (\bar{Q}_{ij})_{90^o}[3^3 - 0] \}$$

$$= 9 * ((\bar{Q}_{ij})_{0^o} + (\bar{Q}_{ij})_{90^o})$$

Hence, $D_{11} = 9 * (130 + 10) = 1260$

Thus,

$$[D] = \begin{bmatrix} 126 & 45 & 0 \\ 45 & 126 & 0 \\ 0 & 0 & 63 \end{bmatrix} \times 10^9 \; Nm \qquad (4.62)$$

From the A,B, and D matrices, it is clear that there is no coupling between extensional and shear as well as extensional and twisting since, $A_{16} = A_{26} = D_{16} = D_{26} = 0$. Moreover, the bending and twisting coupling is also not present.

Problem 4.2: Find A, B, and D matrices for the 3-ply laminate as shown in the figure. Assume stiffness matrix Q as follows:

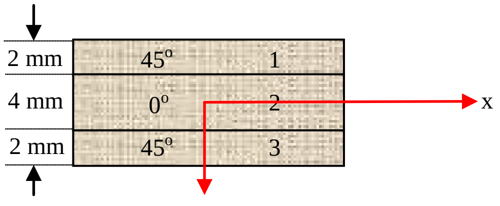

fig.(iii) showing the laminate orientation

For layers 1 and 3 For layer 2

$$[Q]0 = \begin{bmatrix} 2 & 1.5 & 0 \\ 1.5 & 5 & 0 \\ 0 & 0 & 1.5 \end{bmatrix} \text{GPa} \qquad [Q]0 = \begin{bmatrix} 13 & 2.5 & 0 \\ 2.5 & 1 & 0 \\ 0 & 0 & 3.5 \end{bmatrix} GPa \tag{4.63}$$

Solution:

As the laminate is symmetric about its mid-plane, it will have A and D matrices only. The B matrix is zero, i.e. there is no bending and extensional coupling.

The stiffness matrices corresponding to each lamina are to be obtained.

For the middle layer (0^o),

$$[Q]0 = \begin{bmatrix} 13 & 2.5 & 0 \\ 2.5 & 1 & 0 \\ 0 & 0 & 3.5 \end{bmatrix} \text{GPa}$$

For the top and bottom layers (45^o) , the stiffness matrix is obtained by using the expressions for the transformed stiffness matrix,

$$[Q]0 = \begin{bmatrix} 2 & 1.5 & 0 \\ 1.5 & 5 & 0 \\ 0 & 0 & 1.5 \end{bmatrix} \text{GPa}$$

$$\overline{Q_{11}} = Q_{11}\cos^4\theta + Q_{22}\sin^4\theta + 2\left(Q_{12} + 2Q_{66}\right)\sin^2\theta\cos^2\theta$$

$$\overline{Q_{22}} = Q_{11}\sin^4\theta + Q_{22}\cos^4\theta + 2\left(Q_{12} + 2Q_{66}\right)\sin^2\theta\cos^2\theta$$

$$\overline{Q_{12}} = \left(Q_{11} + Q_{22} - 4Q_{66}\right)\sin^2\theta\cos^2\theta + Q_{12}\left(\sin^4\theta + \cos^4\theta\right)$$

$$\overline{Q}_{16} = \left(Q_{11} - Q_{12} - 2Q_{66}\right)\sin\theta\cos^3\theta - \left(Q_{22} - Q_{12} - 2Q_{66}\right)\sin^3\theta\cos\theta$$

$$\overline{Q}_{26} = \left(Q_{11} - Q_{12} - 2Q_{66}\right)\sin^3\theta\cos\theta - \left(Q_{22} - Q_{12} - 2Q_{66}\right)\sin\theta\cos^3\theta$$

$$\overline{Q_{66}} = \left(Q_{11} + Q_{22} - 2Q_{12} - 2Q_{66}\right)\sin^2\theta\cos^2\theta + Q_{66}\left(\sin^4\theta + \cos^4\theta\right)$$

Thus, $\bar{Q}_{11} = 20*\cos^4 45^o + 5 * \sin^4 45^o + 2*(1.5+ 2*1.5) \sin^2 45^o *\cos^2 45^o \quad = 8.5$ (4.64)

Therefore, $$\left[\bar{Q}\right]45 = \begin{bmatrix} 8.5 & 4.84 & 3.75 \\ 4.84 & 8.5 & 3.75 \\ 3.75 & 3.75 & 5.5 \end{bmatrix} \text{GPa}$$

The mid-plane is identified and the co-ordinates for each lamina is calculated and are given below.

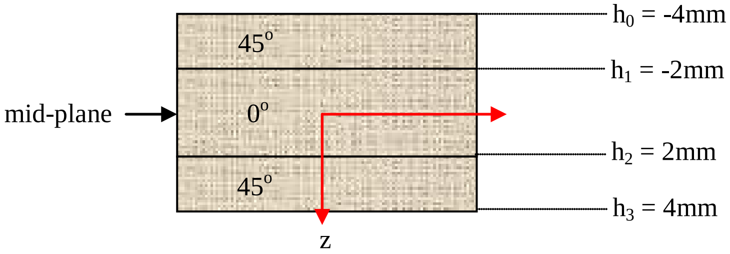

fig.(iv) showing the laminate orientation

[A] matrix is determined by using the equation,

$$\left[A_{ij}\right] = \sum_{k=1}^{n} \left(\bar{Q}_{ij}\right)_k \left(h_k - h_{k-1}\right)$$

n = 3 layers

$$= (\bar{Q}_{ij})_{45^o}[-2 - (-4)] + (\bar{Q}_{ij})_{0^o}[2 - (-2)] + (\bar{Q}_{ij})_{45^o}[4 - 2]$$

$$= 2*(\bar{Q}_{ij})_{45^o} + 4*(\bar{Q}_{ij})_{0^o} + 2*(\bar{Q}_{ij})_{45^o}$$

$$= 4*((\bar{Q}_{ij})_{45^o} + (\bar{Q}_{ij})_{0^o})$$

Hence, $A_{11} = 4 * ((\bar{Q}_{11})_{45^o} + (\bar{Q}_{11})_{0^o}) = 4 * (8.5 + 130) = 554$

Thus,

$$[A]=\begin{bmatrix}554 & 29.36 & 15\\ 29.36 & 74 & 15\\ 15 & 15 & 36\end{bmatrix}\frac{\times 10^9\ N}{m} \tag{4.65}$$

[B] matrix is zero due to laminate symmetry

$$[B] = 0$$

[D] matrix is determined using the equation,

$$\left[D_{ij}\right]=\frac{1}{3}\sum_{k=1}^{n}\left(\bar{Q}_{ij}\right)_k\left(h_k^3-h_{k-1}^3\right) \tag{4.66}$$

$$= ⅓\ \{\ (\bar{Q}_{ij})_{45}{}^{o}[(-2)^3 - (-4)^3] + (\bar{Q}_{ij})_{0}{}^{o}[\ 2^3 - (-2)^3]\ \ + (\bar{Q}_{ij})_{45}{}^{o}[4^3 - 2^3]\ \}$$

$$=\ (\ 112/3\ ) * (\bar{Q}_{ij})_{45}{}^{o} + (16/3) * (\bar{Q}_{ij})_{0}{}^{o}$$

Hence, $D_{11}\ =\ (112/3) * 8.5 + (16/3) * 130 = 1010.67$

Thus,

$$[D]=\begin{bmatrix}1010.67 & 194.03 & 14\\ 194.03 & 370.67 & 14\\ 14 & 14 & 224\end{bmatrix}\times 10^9\ \text{Nm} \tag{4.67}$$

Problem 4.3: Calculate lamina stress variation in each lamina in the laminate given in the Problem 4.1. The load applied is N_x = 100 kN/m.

Solution:

From problem 4.1,

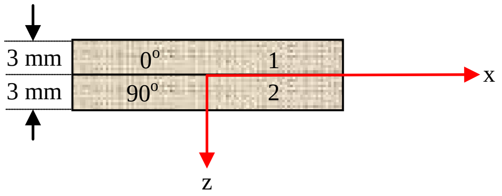

Figure (v) showing the laminate orientation

$$[A]=\begin{bmatrix}42 & 15 & 0\\ 15 & 42 & 0\\ 0 & 0 & 21\end{bmatrix}\frac{\times 10^9\ N}{m} \tag{4.68}$$

$$[B]=\begin{bmatrix}-54 & 0 & 0\\ 0 & 54 & 0\\ 0 & 0 & 0\end{bmatrix}\times 10^9\ N \tag{4.69}$$

$$[D]=\begin{bmatrix}126 & 45 & 0\\ 45 & 126 & 0\\ 0 & 0 & 63\end{bmatrix}\times 10^9\ Nm \tag{4.70}$$

The mid-plane strains and curvature may be calculated using,

$$\{\varepsilon_0\}=[A']\{N\}+[B']\{M\}$$

$$\{k\}=[C']\{N\}+[D']\{M\} \tag{4.71}$$

But, no moment is acting on the given laminate, i.e. {M} = 0. Therefore,

$$\{\varepsilon_0\}=[A']\{N\}$$

$$\{k\}=[C']\{N\} \tag{4.72}$$

where, $[A'] = [A^{-1}] + [A^{-1}][B][(D^*)^{-1}][B][A^{-1}]$

$[C'] = -[(D^*)^{-1}][B][A^{-1}]$

$[D^*] = [D] - [B][A^{-1}][B]$

First calculate the required matrices,

$$\left[A^{-1}\right]=\begin{bmatrix}2.384\times10^{-3} & -8.514\times10^{-5} & 0\\ -8.514\times10^{-5} & 2.384\times10^{-3} & 0\\ 0 & 0 & 0.0476\end{bmatrix}_{\times10^{-9}\ \mathrm{N/m}} \tag{4.73}$$

$$[B]\left[A^{-1}\right][B]=\begin{bmatrix}695.17 & 24.83 & 0\\ 24.83 & 695.17 & 0\\ 0 & 0 & 0\end{bmatrix}_{\times10^{9}\ Nm} \tag{4.74}$$

$[D^*] = [D] - [B][A^{-1}][B]$

$$=\begin{bmatrix} 564.83 & 20.17 & 0 \\ 20.17 & 564.83 & 0 \\ 0 & 0 & 63 \end{bmatrix}_{\times 10^{9}\ \mathrm{Nm}} \tag{4.75}$$

$$=\begin{bmatrix} 564.83 & 20.17 & 0 \\ 20.17 & 564.83 & 0 \\ 0 & 0 & 63 \end{bmatrix} \mathrm{x}10^{9}\ \mathrm{Nm}$$

$$[A'] = [A^{-1}] + [A^{-1}][B][(D^{*})^{-1}][B][A^{-1}]$$

$$=\begin{bmatrix} 5.318\times10^{-3} & -1.899\times10^{-4} & 0 \\ -1.899\times10^{-4} & 5.318\times10^{-3} & 0 \\ 0 & 0 & 0.0476 \end{bmatrix}_{\times 10^{-9}\ m/N} \tag{4.76}$$

$$[C'] = -[(D^{*})^{-1}][B][A^{-1}]$$

$$=\begin{bmatrix} 2.279\times10^{-3} & -7.366\times10^{-9} & 0 \\ -7.366\times10^{-9} & 2.279\times10^{-3} & 0 \\ 0 & 0 & 0 \end{bmatrix}_{\times 10^{-9}\ 1/\mathrm{N}}$$

Therefore,

$$\{\varepsilon_o\} = [A'] \{N\} \tag{4.77}$$

$$=\begin{bmatrix} 5.318\times10^{-3} & -1.899\times10^{-4} & 0 \\ -1.899\times10^{-4} & 5.318\times10^{-3} & 0 \\ 0 & 0 & 0.0476 \end{bmatrix} \times10^{-9} \begin{Bmatrix} 100\times10^{3} \\ 0 \\ 0 \end{Bmatrix}$$

$$=\begin{Bmatrix} 5.318\times10^{-7} \\ -1.899\times10^{-8} \\ 0 \end{Bmatrix}_{m/m}$$

$$\{k\} = [C'] \{N\} \tag{4.78}$$

$$=\begin{bmatrix} 2.279\times10^{-3} & -7.366\times10^{-9} & 0 \\ -7.366\times10^{-9} & 2.279\times10^{-3} & 0 \\ 0 & 0 & 0 \end{bmatrix} \times10^{-9} \begin{Bmatrix} 100\times10^{3} \\ 0 \\ 0 \end{Bmatrix}$$

$$= \begin{Bmatrix} 2.279 \times 10^{-7} \\ -7.366 \times 10^{-13} \\ 0 \end{Bmatrix}_{1/m}$$

The strain is calculated by the following equation,

$$\{\varepsilon\} = \{\varepsilon_0\} + z\,\{k\} \tag{4.79}$$

$$\begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{Xy} \end{Bmatrix} = \begin{Bmatrix} 5.318 \times 10^{-7} \\ -1.899 \times 10^{-8} \\ 0 \end{Bmatrix} + z \begin{Bmatrix} 2.279 \times 10^{-7} \\ -7.366 \times 10^{-13} \\ 0 \end{Bmatrix}$$

In 0^{o} layer,

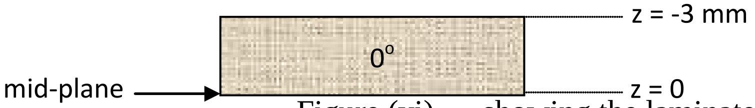

Figure (vi) showing the laminate orientation

Therefore,

$$\begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{Xy} \end{Bmatrix}_{z=-3mm} = \begin{Bmatrix} -1.519 \times 10^{-7} \\ -1.899 \times 10^{-8} \\ 0 \end{Bmatrix} \tag{4.80}$$

$$\begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{Xy} \end{Bmatrix}_{z=0} = \begin{Bmatrix} 5.318 \times 10^{-7} \\ -1.899 \times 10^{-8} \\ 0 \end{Bmatrix}$$

In 90^{o} layer,

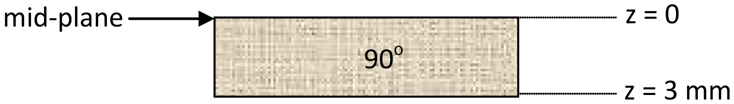

Figure (vii) showing the laminate orientation

Therefore,

$$\begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{Xy} \end{Bmatrix}_{z=0} = \begin{Bmatrix} 5.318 \times 10^{-7} \\ -1.899 \times 10^{-8} \\ 0 \end{Bmatrix} \tag{4.81}$$

$$\begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{Xy} \end{Bmatrix}_{z=3\,mm} = \begin{Bmatrix} 1.216\times10^{-6} \\ -1.899\times10^{-8} \\ 0 \end{Bmatrix}$$

The stresses in each lamina are calculated by using the equation,

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix} = [\bar{Q}] \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix} \tag{4.82}$$

For 0^{o} layer,

$$\left[\bar{Q}\right]0 = \begin{bmatrix} 13 & 2.5 & 0 \\ 2.5 & 10 & 0 \\ 0 & 0 & 3.5 \end{bmatrix} \text{GPa}$$

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix}_{z=-3\,mm} = \begin{bmatrix} 13 & 2.5 & 0 \\ 2.5 & 10 & 0 \\ 0 & 0 & 3.5 \end{bmatrix} GPa \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix}_{z=-3\,mm} \tag{4.83}$$

$$= \begin{bmatrix} -19.794 \\ -0.570 \\ 0 \end{bmatrix} \text{kN/m}^2$$

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix}_{z=0\,mm} = \begin{bmatrix} 13 & 2.5 & 0 \\ 2.5 & 10 & 0 \\ 0 & 0 & 3.5 \end{bmatrix} GPa \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix}_{z=0} \tag{4.84}$$

$$= \begin{bmatrix} 69.087 \\ 1.140 \\ 0 \end{bmatrix} \text{kN/m}^2$$

For 90^o layer,

$$\left[\bar{Q}\right]90 = \begin{bmatrix} 1 & 2.5 & 0 \\ 2.5 & 13 & 0 \\ 0 & 0 & 3.5 \end{bmatrix} \text{GPa}$$

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix}_{z=0} = \begin{bmatrix} 1 & 2.5 & 0 \\ 2.5 & 13 & 0 \\ 0 & 0 & 3.5 \end{bmatrix} GPa \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix}_{z=0}$$

$$= \begin{bmatrix} 5.271 \\ -1.140 \\ 0 \end{bmatrix} \text{kN/m}^2$$

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix}_{z=3\,mm} = \begin{bmatrix} 10 & 2.5 & 0 \\ 2.5 & 13 & 0 \\ 0 & 0 & 3.5 \end{bmatrix} GPa \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix}_{z=3\,mm}$$

$$= \begin{bmatrix} 12.113 \\ 0.570 \\ 0 \end{bmatrix} \text{kN/m}^2$$

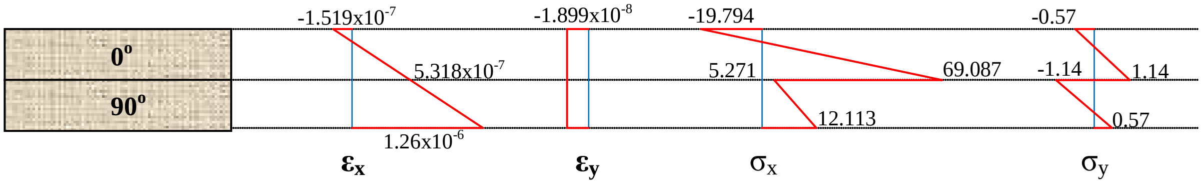

Figure (viii) showing the laminate orientation

stress and strain variations across the thickness